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# Projection Operator Strategies in the Optimization of Trajectory Functionals

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The PROjection operator based Newton method for Trajectory Optimization (PRONTO) is an iterative algorithm which, in its simplest form, allows one to perform local Newton (or quasi-Newton) optimization of the cost functional

$$h((x(\cdot), u(\cdot))) := \int_0^{t_f} l(x(\tau), u(\tau), \tau) d\tau + m(x(t_f))$$

over the set of trajectories of a nonlinear system  $\dot{x} = f(x, u)$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , subject to a fixed initial condition  $x_0$ . Here, we use the word *trajectory* in an extended sense to indicate the state-control pair  $\eta(t) = (x(t), u(t))$ ,  $t \geq 0$ , that satisfies  $\dot{x}(t) = f(x(t), u(t))$  for all  $t \geq 0$ . As usual, “all  $t$ ” means “almost all  $t$ ” in the sense that  $x(t) = x(0) + \int_0^t f(x(\tau), u(\tau)) d\tau$  where  $\int \dots d\tau$  is the Lebesgue integral. The cost functional  $h$  above, defined in terms of the incremental and terminal costs  $l(\cdot, \cdot)$  and  $m(\cdot)$ , and the control vector field  $f$  are taken to be sufficiently smooth ( $C^2$  in  $(x, u)$  and continuous in  $t$  is usually enough) and regular [3].

As shown in [4], the set  $\mathcal{T}$  of trajectories of the nonlinear control system  $\dot{x} = f(x, u)$  has the structure of a (infinite dimensional) Banach manifold, a fact that allows one to use vector space operations [9] to effectively explore it. To work on the trajectory manifold  $\mathcal{T}$ , one *projects* state-control curves in the ambient Banach space onto  $\mathcal{T}$  by using a local linear time-varying trajectory tracking controller. To this end, suppose that  $\xi = (\alpha(\cdot), \mu(\cdot))$  is a bounded state-control curve (an approximate trajectory) and let  $\eta = (x(\cdot), u(\cdot))$  be the trajectory of  $\dot{x} = f(x, u)$  determined by the nonlinear feedback system

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), \\ u(t) &= \mu(t) + K(t)(\alpha(t) - x(t)),\end{aligned}$$

with  $x(0) = x_0$ . Under the hypotheses that the control vector field  $f$  is  $C^r$  and the gain  $K$  is bounded [4], this feedback system defines a  $C^r$  nonlinear operator

$$\mathcal{P} : \xi = (\alpha(\cdot), \mu(\cdot)) \mapsto \eta = (x(\cdot), u(\cdot)) .$$

It is straightforward to see that  $\xi$  is a fixed point of  $\mathcal{P}$ ,  $\xi = \mathcal{P}(\xi)$ , if and only if  $\xi$  is a trajectory of the control system  $\dot{x} = f(x, u)$ . This ensures that  $\mathcal{P}^2 = \mathcal{P}$  so that  $\mathcal{P}$  is a *projection operator*. With this projection operator at hand, one can see [3] that the constrained and unconstrained optimization problems

$$\min_{\xi \in \mathcal{T}} h(\xi) \quad \text{and} \quad \min_{\xi} h(\mathcal{P}(\xi))$$

are essentially equivalent in the sense that a solution to the first *constrained* problem is a solution to the second *unconstrained* problem, while a solution to the second problem is, projected by  $\mathcal{P}$ , a solution to the first problem. Using these facts, one may develop Newton and quasi-Newton descent methods for trajectory optimization in an effectively unconstrained manner by working with the cost functional  $g(\xi) := h(\mathcal{P}(\xi))$ .

The Projection Operator based Newton method for Trajectory Optimization (PRONTO) is given by [3]

**Algorithm** (Projection operator Newton method)

**given** initial trajectory  $\xi_0 \in \mathcal{T}$

**for**  $i = 0, 1, 2, \dots$

redesign feedback  $K$  if desired/needed

$$\zeta_i = \arg \min_{\zeta \in T_{\xi_i} \mathcal{T}} Dh(\xi_i) \cdot \zeta + \frac{1}{2} D^2 g(\xi_i) \cdot (\zeta, \zeta) \quad (\text{search direction})$$

$$\gamma_i = \arg \min_{\gamma \in (0,1]} g(\xi_i + \gamma \zeta_i) \quad (\text{step size})$$

$$\xi_{i+1} = \mathcal{P}(\xi_i + \gamma_i \zeta_i) \quad (\text{update})$$

**end**

Note that the functional  $g(\cdot)$  and the projection operator  $\mathcal{P}$  depend on the choice of the feedback  $K$ . Also,  $Dg(\xi_i)$  and  $D^2 g(\xi_i)$  are the first and second Fréchet derivatives of the Banach space functional  $g$ . When  $\xi \in \mathcal{T}$  and  $\zeta \in T_\xi \mathcal{T}$ , the first derivative  $Dg(\xi) \cdot \zeta$  simply equals  $Dh(\xi) \cdot \zeta$ , i.e., it does *not* depend on  $\mathcal{P}$ .

At each step, the minimization of a second order approximation of the extended cost functional  $g$  provides a *search direction*. Then an optimal *step size* is computed through a (backtracking) line search (a pure Newton method would use a fixed step size of  $\gamma_i = 1$ ). Combining the search direction  $\zeta_i$  with step size  $\gamma_i$  a new *update* trajectory is computed and the algorithm restarts (unless a termination condition is met). An illustration of the projection operator approach is shown in Figure 1.

The computed optimal search direction  $\zeta_i$  is constrained to lie on the tangent space to the trajectory manifold at the current iterate, i.e.,  $\zeta_i \in T_{\xi_i} \mathcal{T}$ . This is not restrictive since, as established in [4, Proposition 3.2],  $\mathcal{P}$  can be used to define a bijection between the neighborhood of a trajectory  $\xi \in \mathcal{T}$  and the origin of its tangent space  $T_\xi \mathcal{T}$ . The condition  $\zeta_i \in T_{\xi_i} \mathcal{T}$  simply means that  $\zeta_i(t) := (z_i(t), v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^m$ ,  $t \geq 0$ , is a trajectory of the linearization of the control system  $\dot{x} = f(x, u)$  about the current trajectory iterate  $\xi_i$ . The search direction subproblem is, in practice, a linear quadratic (LQ) optimal control problem, where the functional to be minimized,  $Dh(\xi_i) \cdot \zeta + \frac{1}{2} D^2 g(\xi_i) \cdot (\zeta, \zeta)$ , is the quadratic model functional given by the first two terms of the Taylor expansion of the functional  $g(\xi_i + \zeta)$  with respect to  $\zeta$  [3, Section 3]. The LQ problem is defined using first and second order derivatives of the nonlinear

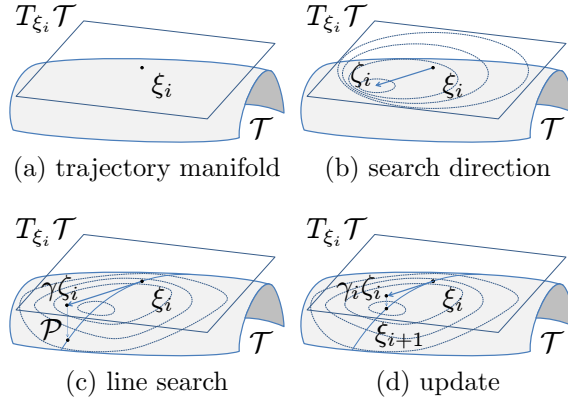


Figure 1: The projection operator approach; (a) at each iteration, the linearization of the control system about the trajectory  $\xi_i$  defines the tangent space to the trajectory manifold  $\mathcal{T}$  at  $\xi_i$ ; (b) the constrained minimization over the tangent space of the second order approximation of the extended cost functional  $g = h \circ \mathcal{P}$  yields the *search direction*  $\zeta_i$ ; (c) the optimal *step size* is computed through a line search along  $\zeta_i$ ; (d) the search direction  $\zeta_i$  and step size  $\gamma_i$  are combined to obtain a new *update* trajectory  $\xi_{i+1}$ .

system and the incremental and terminal costs about the current (nonlinear system) trajectory iterate. It can be solved by computing the solution to a suitable differential Riccati equation (and an associated adjoint system). In particular, in the vector space case, the usual chain rule applies and one finds that  $D^2g(\xi) \cdot (\zeta, \zeta)$  is a well defined object given by

$$D^2g(\xi) \cdot (\zeta, \zeta) = D^2h(\xi) \cdot (\zeta, \zeta) + Dh(\xi) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta, \zeta), \quad (1)$$

for  $\xi \in \mathcal{T}$  and  $\zeta \in T_\xi \mathcal{T}$  [4]. Note that  $D^2\mathcal{P}(\xi)$  is the second Fréchet derivative of the Banach space *operator*  $\mathcal{P}$ .

Our work has been focused on understanding the nature of highly nonlinear dynamic systems and especially those with significant maneuvering objectives. We believe that trajectory optimization provides strong tools and techniques for discovering and understanding important dynamic features for a broad range of systems. We also believe that it is only by doing significant numerical exploration on difficult nonlinear systems that we begin to understand *how* such explorations may be accomplished through the use of appropriate models for the systems together with appropriate cost objectives and constraints. To this end we have worked with systems ranging from classical nonlinear pendulum systems [1, 5] to air [12, 15, 11], land [10, 23, 13, 2, 14, 24, 25], and marine [7] vehicles, and even earthquake shakatables [6]. Since many systems of interest do not evolve in a *flat* space, we have devoted significant effort to understanding and extending the projection operator approach to work with manifold and especially with Lie groups [18, 19, 20, 21, 16, 22, 17].

Throughout this work, we have developed important insights into the nature of trajectory functionals and their minimization. Second order properties of the trajectory functionals have helped us to discover cases where one would expect a functional to have a nice minimizer but in fact one does not exist. Barrier functionals have been used to effectively manage input, state, and mixed constraints. Working with such a variety of systems and objectives, we have developed a sort-of experts toolkit that makes such investigations (somewhat) more tractable. We have found that, in order to do effective trajectory exploration and optimization, one must indeed become an *expert* on the system under investigation!

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